Introduction

For the sake of continuity, we give a brief background of the difficulties in numerical evaluation of Hankel transform and construct an efficient and stable algorithm for its numerical evaluation and apply it to solve the realistic problem of inverse quasi-static steady-state thermal stresses in a thick circular plate, which is subjected to arbitrary interior temperature and determine the unknown temperature and thermal stresses.

The Hankel Transform

The general Hankel transform pair for Bessel function of order ν is defined as [1,2]

$$H_\nu[f(r); p] = \int_0^\infty r f(r) J_\nu(pr) dr = F_\nu(p). \tag{1}$$

Hankel Transform is self reciprocal; its inverse is given by

$$H_\nu^{-1}[F_\nu(p); r] = \int_0^\infty p F_\nu(p) J_\nu(pr) dp = f(r), \tag{2}$$

where J_ν is the ν th-order Bessel function of first kind.

Several quality research articles have been published for the evaluation of Hankel transform. Analytical evaluations of (1) and (2) are rare and their numerical computations are difficult because of the oscillatory behaviour of the Bessel functions and infinite length of the interval involved in it. The efficiency of a method for computing HT is highly dependent on the function to be transformed and thus it is very difficult to choose an optimal algorithm for given function.

Postnikov [3], proposed for the first time, a novel and powerful method for computing zero and first order HT by using Haar wavelets. Refining the idea of Postnikov [3], Singh et al. [4–6] obtained three efficient algorithms for numerical evaluation of HT of order $\nu > -1$. All these algorithms depend on separating the integrand $rf(r) J_\nu(pr)$ into two components; the slowly varying components $rf(r)$ and the rapidly oscillating component $J_\nu(pr)$. Then either $rf(r)$ is expanded into various wavelet series using different orthonormal bases like Haar wavelets, linear Legendre multiwavelets, Fourier Bessel series and truncating the series at an optimal level or approximating $rf(r)$ by a quadratic over the subinterval using the Filon quadrature philosophy [7].

In this manuscript, we take an entirely different approach. Instead of manipulating the simpler component $rf(r)$, we manipulate the rapidly oscillating part $J_\nu(pr)$, thus avoiding the complexity of evaluating integrals involving Bessel functions. We use the hat basis functions described in "Hat Functions and Their Associated Properties" section, to approximate $J_\nu(pr)$ and replace it by its approximation in Eq. (1), thereby getting an efficient and stable algorithm for the numerical evaluation of the HT of order $\nu > -1$. In "Algorithm" section, we derive the algorithm and further give the error and the stability analysis in "Error and Stability Analysis" section. A numerical experiment to verify our theoretical findings is also provided in "Error and Stability Analysis" section. In "The Main Result" section, we apply our proposed algorithm to solve the realistic problem of inverse quasi-static steady-state thermal stresses in a thick circular plate, which is subjected to arbitrary interior temperature and determine the unknown temperature and thermal stresses on the upper surface of the thick circular plate, where the fixed circular edge and the lower surface of the circular plate are thermally insulated.

Hat Functions and Their Associated Properties

Hat functions are defined on the domain [0, 1] . These are continuous functions with shape of hats, when plotted on two dimensional plane. The interval [0, 1] is divided into n subintervals $[ih, (i+1)h]$, $i = 0, 1, 2, \ldots, n-1$, of equal lengths h where $h = \frac{1}{n}$. The hat function's family of first $(n+1)$ hat functions is defined as follows [8]:

$$\psi_0(t) = \begin{cases} \frac{h-t}{h}, & 0 \le t < h, \\ 0, & otherwise, \end{cases} \tag{3}$$

$$\psi_i(t) = \begin{cases} \frac{t-(i-1)h}{h}, & (i-1)h \le t < ih, \\ \frac{(i+1)h-t}{h}, & ih \le t < (i+1)h, i = 1, 2, \ldots, n-1, \\ 0, & otherwise, \end{cases} \tag{4}$$

$$\psi_n(t) = \begin{cases} \frac{t-(1-h)}{h}, & 1-h \le t \le 1, \\ 0, & otherwise. \end{cases} \tag{5}$$

From the definition of hat functions it is obvious that

$$\psi_i(kh) = \begin{cases} 1, & i = k, \\ 0, & i \ne k, \end{cases} \tag{6}$$

The hat functions $\psi_j(t)$ are continuous, linearly independent and are in $L^2[0, 1]$.

A function $f \in L^2[0, 1]$ may be approximated as

$$f(t) \simeq \sum_{i=0}^{i=n} f_i \psi_i(t) = f_0\psi_0(t) + f_1\psi_1(t) + f_2\psi_2(t) + \cdots + f_n\psi_n(t). \tag{7}$$

The important aspect of using extended hat functions in the approximation of function $f(t)$, lies in the fact that the coefficients f_i in the Eq. (7), are given by

$$f_i = f(ih), \quad \text{for } i = 0, 1, 2 \ldots, n \quad \text{where } h = 1/n. \tag{8}$$

Algorithm

To derive the algorithm, we first assume that the domain space of input signal $f(r)$ extends over a limited region $0 \le r \le R$. From physical point of view, this assumption is reasonable due to the fact that the input signal $f(r)$ which represents the physical field is either zero or it has an infinitely long decaying tail out side a disc of finite radius-R. Therefore, in many practical applications either the input signal $f(r)$ has a compact support or for a given $\varepsilon > 0$ there exists a positive real R such that $\left|\int_R^\infty rf(r)J_\nu(pr)dr\right| < \varepsilon$, which is the case if $f(r) = o(r^\lambda)$, where $\lambda < -\frac{3}{2}$ as $r \to \infty$. Hence in either case, from Eq. (1), we have

$$\hat{H}_\nu[f(r); p] = \int_0^R rf(r)J_\nu(pr)dr \equiv \hat{F}_\nu(p). \tag{9}$$

By scaling (9) may be written as

$$\hat{F}_\nu(p) = \int\limits_0^1 r f(r) J_\nu(pr) dr, \tag{10}$$

which is known as finite Hankel transform (FHT). Equation (9) is a good approximation of the HT given by (2). Using Eqs. (7) and (8), $J_\nu(pr)$ may be approximated as

$$J_\nu(pr) \simeq \sum_{i=0}^{n} J_\nu(pih)\psi_i(r). \tag{11}$$

Using the approximation in (11), we get the algorithm to evaluate the Hankel transform as

$$\begin{aligned}
\hat{F}_\nu(p) &\simeq \int\limits_0^1 r f(r) \sum_{i=0}^{n} J_\nu(pih)\psi_i(r) dr \\
&= \sum_{i=0}^{n} J_\nu(pih) \int\limits_0^1 r f(r)\psi_i(r) dr \\
&= J_\nu(0) \int\limits_0^h r f(r)\psi_0(r) dr + \sum_{i=1}^{n-1} J_\nu(pih) \int\limits_{(i-1)h}^{(i+1)h} r f(r)\psi_i(r) dr \\
&\quad + J_\nu(p) \int\limits_{1-h}^{1} r f(r)\psi_n(r) dr.
\end{aligned} \tag{12}$$

It is note worthy here that the integral $\int_0^1 r f(r)\psi_i(r) dr$ appearing in Eq. (12), may be easily calculated as $f(r)$ is known function and $\psi_i(r)$ is a linear polynomial $\forall i$.

Error and Stability Analysis

Let the R.H.S. of (11) is denoted by $J_{\nu,n}(pr)$ i.e.

$$J_{\nu,n}(pr) = \sum_{i=0}^{n} J_\nu(pih)\psi_i(r). \tag{13}$$

Now replacing $J_\nu(pr)$ in Eq. (10), we define an nth approximate $\hat{F}_{\nu,n}(p)$ of the FHT $\hat{F}_\nu(p)$ as follows:

Definition 4.1 An nth approximate finite Hankel transform of $f(r)$, denoted by $\hat{F}_{\nu,n}(p)$ is defined as

$$\hat{F}_{\nu,n}(p) = \int_0^1 r f(r) J_{\nu,n}(pr) dr = \sum_{i=0}^{n} J_\nu(pih) \int_0^1 r f(r)\psi_i(r) dr. \tag{14}$$

Let $\varepsilon_n(p)$ denote the absolute error between the FHT $\hat{F}_\nu(p)$ and its nth approximate $\hat{F}_{\nu,n}(p)$, then we have the following:

Theorem 4.1 *If $J_\nu(pr)$ is approximated by the family of first $(n+1)$ hat functions as given in Eq. (11), then*

(i) $\left|J_\nu(pjh) - J_{\nu,n}(pjh)\right| = 0,$ *for* $j = 0, 1, 2, \ldots, n.$

(ii) $\left|J_\nu(pr) - J_{\nu,n}(pr)\right| \leq \frac{p^2}{2n^2} + O\left(\frac{p^3}{n^3}\right),$ *for* $jh < r < (j+1)h,\ j = 0, 1, 2, \ldots, n-1.$

(iii) $\varepsilon_n(p) = \left|\hat{F}_\nu(p) - \hat{F}_{\nu,n}(p)\right| \leq \frac{Mp^2}{4n^2} + O\left(\frac{p^3}{n^3}\right),$ *where* $|f(r)| \leq M.$

Proof See the Appendix 1.

The stability of the proposed algorithm is analyzed under the influence of noise. In what follows, the exact data function is denoted by $f(r)$ and the noisy data function $f^\alpha(r)$ is obtained by adding a random noise α to $f(r)$ such that $f^\alpha(r) = f(r) + \alpha\theta$, where θ is a uniform random variable with values in $[-1, 1]$ such that $|f^\alpha(r) - f(r)| \leq \alpha$. Then we have: □

Theorem 4.2 *When the input signal $f(r)$ is corrupted with noise α, the proposed algorithm reduces the noise at least by a factor of $1/2$ in the output data $\hat{F}_{\nu,n}$.*

Proof See Apendix 2.

A test problem included in this section is solved with and without random perturbations (noises) to illustrate the efficiency and stability of proposed algorithm by choosing three different values of noise α as $\alpha_0 = 0$, $\alpha_1 = 0.002$ and $\alpha_2 = 0.005$.

The errors $E_j(p)$ (= the approximate FHT obtained from Eq. (12) with random noise α_j−the exact FHT), $j = 0, 1, 2$ are computed and their graphs are sketched, for different n. Further the parameter p ranges between 0 to 30 in steps of 0.2. Figure 3 depicts the graph of $\left|\hat{F}^\alpha_{\nu,n}(p) - \hat{F}_{\nu,n}(p)\right|$ for the test function in example, which is in conformity with the Theorem 4.2. For this illustration, the computations are done in MATLAB 7.0.1 and the elapsed times in computations of FHT by CPU for MATLAB codes, are found to be 0.140, 0.593, 4.961 and 69.545 s for n=100, 1000, 10,000 and 100,000 respectively. The least square errors $\left\|E_j(p)\right\|_2$ involved in computations of approximate FHT with noises α_j, $j = 0, 1, 2$ for the given example with n =10,000, are $1.0554E-08$, $1.0554E-08$ and $1.0554E-08$. These are calculated, using the formula

$$\left\|E_j(p)\right\|_2 = \sqrt{\frac{\sum\limits_{i=0}^{n} E_j^2(p_i)}{n+1}},$$

where p_i is taken in steps of 0.2 in the range [0, 30]. □

Example Consider the function $f(r) = \left(r^2 - a^2\right)^2$given in [9], whose zero order finite Hankel transform is given by $F_0(p) = \frac{8a\left\{\left(8-a^2p^2\right)J_1(pa) - 4apJ_0(pa)\right\}}{p^5}$.

For numerical computation, we take $a = 1$ to show comparison between exact HT $F_0(p)$ and nth approximate FHT $\hat{F}_{0,n}(p)$, in Fig. 1. The errors $E_0(p)$,$E_1(p)$ and $E_2(p)$ for $n = 100$ are shown in Fig. 2. Further Fig. 3 depicts the graph of $\left|\hat{F}^\alpha_{0,n}(p) - \hat{F}_{0,n}(p)\right|$, for noises $\alpha = 0.002, 0.005$ and random variable $\theta = 0.2311$. Again it is to be noted here that in the caption of Fig. 3, $\left|\hat{F}^\alpha_{0,n}(p) - \hat{F}_{0,n}(p)\right|$ is denoted by $\left|H^\alpha_0(p) - H_0(p)\right|$.