Chapter 32
The Use of Cubic Equations in Islamic Art and Architecture

Alpay Özdural

Introduction

The predominance of geometry in the ornamental arts that adorn the buildings of the Islamic world, from Spain to Central Asia, has always been a fruitful field of research. Starting in Umayyad times in Damascus as hesitant experiments, simple geometric motifs had developed almost instantly into intriguing and awe-inspiring ornamental geometric compositions that reached continually new climaxes, with increased complexity or new horizons, in Baghdad, Isfahan, Cordoba, Alhambra, Tabriz, Samarqand, Delhi, Istanbul, and again Isfahan. It is generally viewed that those intricate patterns were conceived and produced by artisans who were not only masters in their crafts but also in geometry. To imagine all those medieval artisan/architects also as mathematicians well versed in Euclid, though an attractive thought, had always seemed rather implausible to me since they were known to be mostly illiterate. Lately I have been developing the ideas that most of the aesthetic, structural or spatial innovations that we observe in the major architectural centers of the Islamic world were mainly due to the active role of mathematicians at the conception stage, and that some of the great accomplishments of Islamic art and architecture can be explained as the products of the collaboration between mathematicians and artisans at special meetings.[1]

This sort of collaboration is best exemplified by a Persian work on ornamental geometry, *Fī tadākhul al-ashkāl al-mutashābiha aw al-mutawāfiqa* (*On interlocks*

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Alpay Özdural (1944–2003).

[1] The first part of the argument was first mentioned in (Holod 1988). For my publications on this point, see (Özdural 1995, 1996, 1998, 2000). Similar views are expressed in (Necipoğlu 1995: 167–175).

K. Williams and M.J. Ostwald (eds.), *Architecture and Mathematics from Antiquity to the Future*, DOI 10.1007/978-3-319-00137-1_32,

of similar or complementary figures; referred to hereafter as *Interlocks of Figures*).[2] It is an anonymous work, or rather a collection of 68 separate constructions, which appears to have been compiled from notes taken by a scribe at a series of meetings between mathematicians and artisans around the turn of the fourteenth century. This approximate date corresponds to the golden age of the Ilkhanid era, which began with the reign of Ghazan Khan (1295–1304). He and his vizier, Rashid al-Din (1247–1318), undertook huge construction campaigns around Tabriz and gathered a great number of scholars, scientists, and artisans there. Rashid al-Din specifically dedicated his suburb, the Quarter of Rashid, to the encouragement of the arts and sciences. *Interlocks of Figures* seems to be produced in the vigorous atmosphere created by this intensive architectural activity. The meetings were probably held at Tabriz under the sponsorship of either of the leaders, who perhaps demanded the application of the highest possible advancements in geometry to the ornamental arts.

The last point is my conjecture based on the fact that three of the constructions in *Interlocks of Figures* involve the solutions of cubic equations, one of the great achievements of Islamic mathematics up to that time. These three constructions were essentially verging procedures, that is to say, mechanical equivalents of the solutions by means of conic sections. Owen Jones remarks on the significance of the use of cubic equations in the ornamental arts:

> As with proportion, we think that those proportions will be the most beautiful which it will be most difficult for the eye to detect; so we think that those compositions of curves will be most agreeable, where the mechanical process of describing them shall be least apparent; and shall find it to be universally the case, that in the best periods of art all mouldings and ornaments were founded on curves of higher order, such as the conic sections; whilst, when art declined, circles and compass-work were much more dominant (Jones 1982: 69).

Verging Procedures

The first of these verging procedures surfaces in Construction 16 of *Interlocks of Figures* (Fig. 32.1):

> Triangle AK[B] is a right-angled triangle in which the ratio of the difference between the shortest side and the hypotenuse to the difference between the [former] difference and the shortest side [the rest of the sentence is missing in the text; add "is the same as the ratio of the shortest side to the intermediate side"].
>
> **Section**
>
> The procedure is this:
>
> By means of GD (A, B, D in the text; to make the procedure more understandable, it should read "mark a given length, GD, on an arbitrary line AB, and by means of GD") erect perpendicular GE [equal to GD]. Make point E the center and then with [compass opening] EG describe arc ZH in the direction of B. Bisect GD at point T (E in the text).

[2] The only copy of this manuscript is preserved in Ms. Persan 169 in the Bibliothèque Nationale, Paris, a compilation of twenty-five works on mathematical subjects, mainly practical geometry.

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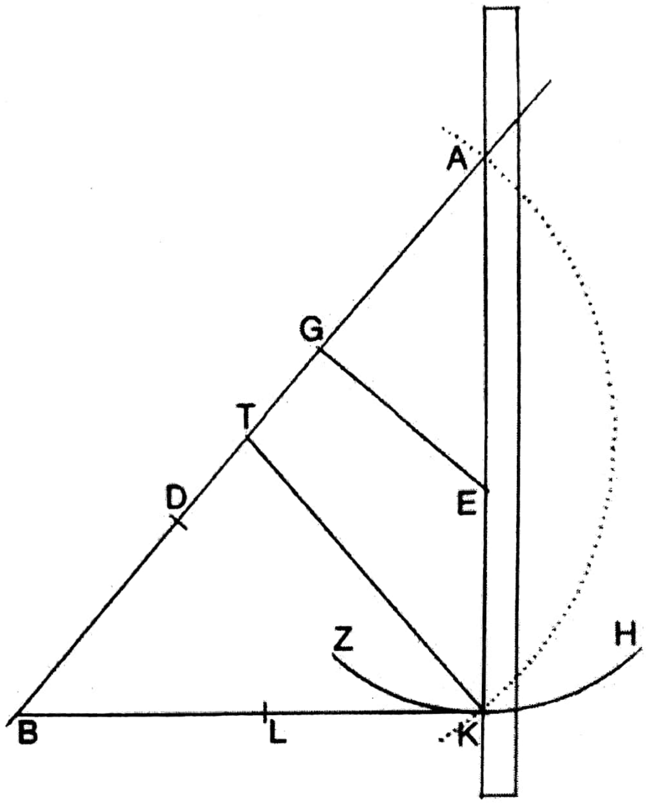


Fig. 32.1 Construction 16 of *Interlocks of Figures*. Image: author

> Put [one] arm of the compass fixed on point [T], and place the straightedge so that its edge always touches point E (more information is needed at this crucial juncture, add "so that at every instance it cuts simultaneously arc ZH at point K and the line at point A. With the other arm of the compass, compare the changing lengths of TK and TA"). Give motion to the straightedge until [the position is reached] that the lengths of TK and T become equal. Mark points K, A, and B [by making TB equal to TA]. When lines AK and KB are drawn, triangle AKB is the required right-angled one. [In this triangle,] AD is equal to BG and BK. When BL, which is equal to BD, is subtracted from BK, KL is equal to GD. God knows best [Bibliothèque Nationale, Paris, Ms. Persan 169, sec. 24, fol. 185r].[3]

The scribe's lack of familiarity with the verging procedure is apparent in the amount of the missing information. He had some acquaintance with ordinary geometrical methods, but when the construction involved advanced techniques, his knowledge proved insufficient to record the explained procedure accurately. Despite all the missing information in the text, we are able to restore it to a fully detailed verging construction. According to the text, the hypotenuse is AB, the intermediate side AK, and the shortest side BK. Also, BK = BG, AG = BD. Then, AB−BK = AG = BD and BG−AG = GD. According to the restored text, AG : GD = AK : KB. If perpendicular GE is drawn, by similar triangles, AG : GE = AK : KB. Then GD = GE. Since angles K and G are right angles, GB = BK, and BE is common, triangles EGB and EKB are congruent. Then GE = EK.

[3] In this translation from the original Persian manuscript, simple restorations are added in square brackets; more detailed ones are explained in parentheses.

The problem in the text is to construct a right-angled triangle ABK with these properties. In other words, to construct triangle ABK so that (AB−BK):(BK−[AB−BK]) = AK:KB. Since GD is assumed in the text as the given length, the problem is to determine points A, K and B. The solution is reached by means of a verging procedure, that is to say, constructing points A and K by way of iteration in such a way that AEK is straight and AT = TK. If it were only this text that accompanies the construction, it would seem merely a geometric problem. Its real objective is stated in the text added to the reverse side of the folio:

Section

In this knot pattern ('aqd) we need a right-angled triangle such that if [a length equal to the shortest side] is cut from the hypotenuse of the triangle towards the shorter side and a perpendicular is erected at the point of cutting, it cuts off the intermediate side at a point where [the distance] from it to the right angle is equal to the perpendicular itself.[4]

Obtaining a triangle of this sort is difficult. It falls outside the Elements of Euclid and concerns the science of conics (makhrūṭāt). If the perpendicular length is assumed, as in this example, the construction is achieved by means of a moving straightedge [Bibliothèque Nationale, Paris, Ms. Persan 169, sec. 24, fol. 185v].

The scribe appears more confident in explaining the properties achieved once the construction is completed than when he was explaining the details of the "moving geometry" (the term used by Muslim mathematicians for verging procedures). He also relates the explicit words of the author of the construction that it cannot be achieved by means of compass and straightedge, the tools of Euclidean geometry, because it concerns "the science of conics," i.e., cubic equations. Indeed it does. It is in fact the solution by means of moving geometry for the equation:[5]

$$x^3 + 2x^2 - 2x - 2 = 0 \quad (\text{if GD} = 1 \text{ and GA} = x).$$

Construction of the Knot Pattern (*'aqd*)

We understand that the purpose of the whole exercise was to create a special knot pattern (*'aqd*, as it was called in those days), from which an ornamental composition would be generated. The basic unit in this pattern is a right-angled triangle of which the hypotenuse is divided into three parts by a medial segment in such a way that if a perpendicular is erected from the end of the segment to the intermediate side, the shorter segment it cuts off from the intermediate side would be equal to the length of the perpendicular, which is also equal to the given medial

[4] In Arabic, '*aqd* literally means knot. Here it is used to mean the unit to be repeated to generate an interlocking ornamental composition. To convey both meanings, I translate it as "knot pattern."

[5] It is assumed in the text that the length of segment GD is 1 and GA is x. Then perpendicular GE = 1, EK = 1, BD = x, BK = BG = $1 + x$, EA = $\sqrt{(1 + x^2)}$. Since AG:GE = AK : KB, we have $x : 1 = [1 + \sqrt{(1 + x^2)}]:(1 + x)$. This equation can be reduced to $x^3 + 2x^2 - 2x - 2 = 0$. The equation has one positive root, $x = 1.1700865$ accurate in seven decimals. I also compute the angles: tan ∠AEG = x, so ∠AEG = ∠B = 49.481553°, ∠A = 40.518447°, ∠AKG = ½∠AEG = 24.740777°.

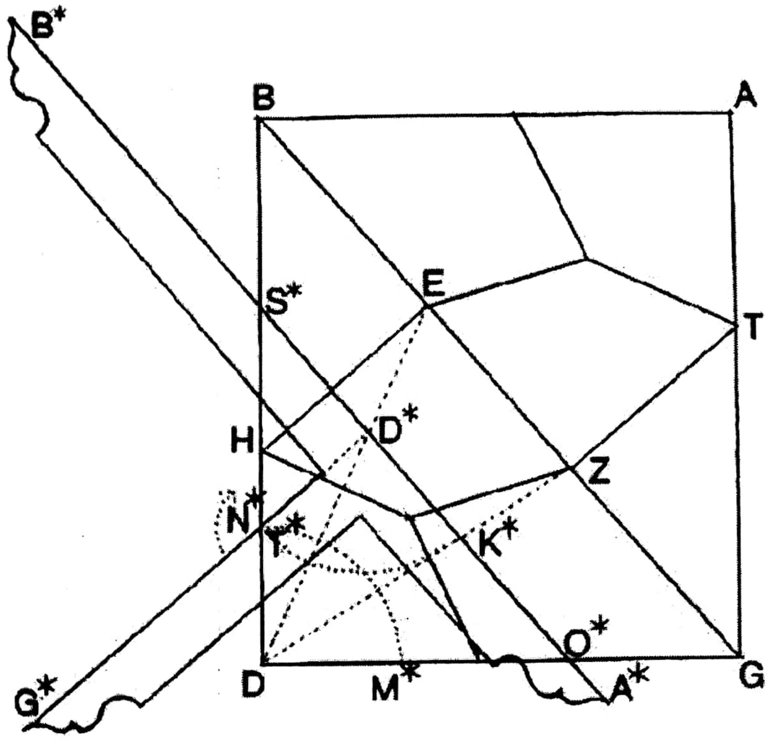


Fig. 32.2 Construction 37 of *Interlocks of Figures*. Image: author

segment (KE = EG = GD). In this seemingly complicated problem, there are two requirements to be met: (1) triangle AKB should be right-angled; (2) the three segments, DG, GE and EK should be equal to each other. Both of these requirements are met simultaneously by means of moving geometry: (1) since TK = TA = TB by the aid of the compass on point T, triangle AKB is half of a rectangle, hence right-angled; (2) since GE is taken equal to GE and arc ZH is drawn with radius EG from center E, EK is equal to EG wherever AEK cuts arc ZH.

One wonders what sort of an ornamental composition would require such delicate properties? It becomes apparent in Construction 37 of *Interlocks of Figures* (Fig. 32.2):

> The construction of this knot pattern is [performed] by a T-ruler (*gunyā mistar*). I say that in this knot pattern defined by the repeat unit (*khāna*, literally "house" or "home" in Persian) ABDG (ABD in the text), it is required that the "orange" ABZT be congruent to the "orange" (*turanj*) DGEH in such a way that BZ will be equal to GE and [a portion of] each will be common to both, thus BE will be equal to GZ. The other [requirement] is that since in the "orange" EGDH sides EG and GD is equal, EH and HD will also be equal. Necessarily, the angles at E and D will be equal and right [angles]. As this preliminary is now known, let us assume that side GD of the knot pattern of the repeat unit is known but the indefinitely extended side DB is unknown (that is, line DB is drawn but point B has not been defined on it).
>
> Then we take the ruler and from point D*, [which marks the intersection of perpendicular arms,] with an arbitrary compass opening mark lengths D*K* and D*T* equal to each other. Then go back to the repeat unit of the knot pattern, and, in the same manner that the lengths are marked on the ruler, mark points M* and N* on sides DG and DB. Take the ruler again and position the letter T* on the perpendicular arm at the letter N* so that both points are fixed on each other. Then give motion to the ruler pivoted on this point from left to right until lengths S*D* and K*O* on either side of the ruler become equal. Point T* should never be

> separated from point N*. [At this position] draw line S*O*. From point G, which is known, and parallel to S*O* draw line GB to define the rectangle. Divide line BG according to the proportion of S*D*, D*K*, and K*O* (it can be done for instance by drawing lines from point D through points D* and K*). On the ruler, which is parallel to GB, length S*D* is equal to K*O*, and K*D* is equal to D*T* and T*D. Thus, on line GB length BE will be equal to ZG, and ZE equal to EH and to HD. These constitute what is required. God knows best [Bibliothèque Nationale, Paris, Ms. Persan 169, sec. 24, fol. 190r].[6]

From the figure and explanation we understand that the gist of this and the previous construction was to create a special rectangular repeat unit so that when two congruent right-angled isosceles quadrangles (which are called "orange" in this example) facing opposite directions are contained in it, the segment that they share on the common diagonal is equal to their shorter sides. What I call "isosceles quadrangle", for want of a better term as it does not exist in modern mathematics, is a figure peculiar to the ornamental arts throughout the Islamic world, but known under different names such as "orange," "pine cone," "almond," "barleycorn" and "rhomboid." In this combined form of two isosceles triangles, with the aid of the axis through their vertices, it can be subdivided into multiple isosceles quadrangles. Depending on the angles of the initial isosceles quadrangle, ornamental configurations can be obtained from these subdivisions when the unit pattern is repeated. It was particularly on this point that the authors of these two constructions had concentrated their efforts. Whichever method of moving geometry is used, the pattern it yields generates by repetition a delicate composition (Fig. 32.3).

The second method too is the mechanical equivalent of a cubic equation, which is the reduced form of the same problem. In this case, since GD $= 1$ and EH $= x$, it corresponds to: $x^3 - 3x^2 - x + 1 = 0$ (Özdural 1996: 199). Here, the procedure of the moving geometry is given in full detail in the text. Apparently this time the scribe was able to record the explanation of the author correctly; but again shows his unfamiliarity with the subject by placing the T-ruler on the wrong side.[7]

In this second solution, the T-ruler has replaced the traditional straightedge in performing the moving geometry. With its permanently perpendicular arms it proves to be a practical and efficient tool in meeting the two requirements of the problem: it ensures that HE is always perpendicular to the diagonal while the movement of its long edge, on which the required proportion of the segments is marked, determines the position of the diagonal. Its useful peculiarities should have

[6] In the figure of the original manuscript, the points that belong to the T-ruler and those used to perform the moving geometry were distinguished by red ink. Some of these were identical to the ones that were used for the pattern itself; and no differentiation was made in the text. In order to avoid the confusion they create, the letters written in red ink are distinguished here by adding stars above, both in the figure and the text.

[7] In the original figure, the T-ruler is placed upon triangle GAB. The explanation in the text, however, makes sense only if the ruler is placed upon triangle GDB.

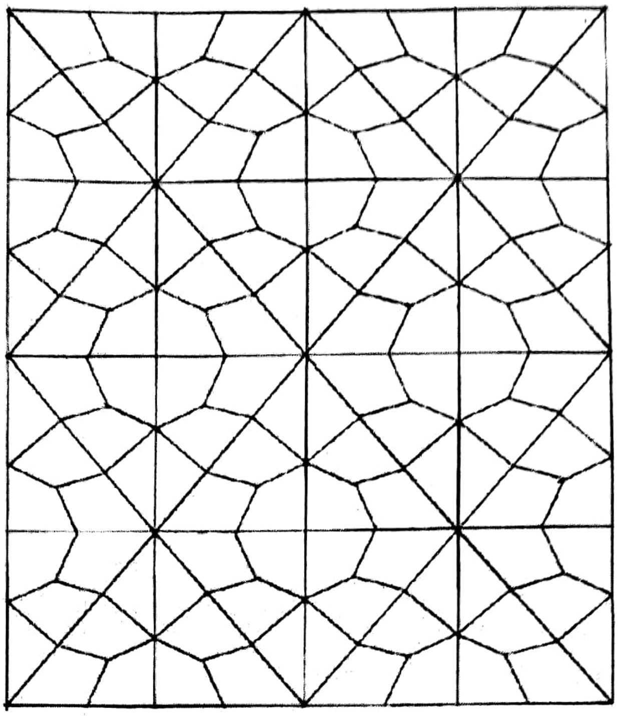


Fig. 32.3 The decorative scheme generated by construction 16 or construction 37. Image: author

attracted the attention of the participants of the discussions; a full page is allocated in *Interlocks of Figures* for its description and potential use (Fig. 32.4):

> The true nature of the proportion of this [preceding] knot pattern belongs to conics. This we can draw with the aid of an instrument called T-ruler. That is an instrument with which many knot patterns formed by conics can be drawn. In fact, this is the opinion of Katib[i]; whether it is true or not is not clear.
>
> Be that as it may, one produces the ruler in the same way as the alidade of an astrolabe (*'iḍada-i asṭurlab*). At the middle of it erect a perpendicular ruler similar to the "arrow" (sahm) of the alidade of the "boat astrolabe" (*asṭurlab-i zawraqī*, which was developed by al-Sijzi ca. 980). This is called the "mast of the bracket" (*saṭāra-i gunyā*).
>
> For example, the ruler ABGD consists of ruler AB and perpendicular [arm] GD. Should an inclination (inhiraf) be given to the edge AB of the ruler, like the inclination of the "tailored (*mujayyab*) alidade," the edge GD of the perpendicular [arm] would have the same declination. While [the declination] from the perpendicular line GD becomes distinctly apparent, point D on the edge of the alidade remains fixed. Angle GDA is found so perfectly perpendicular that with this ruler many amazing proportions (*nisbathā-i gharīb*) can be created [Bibliothèque Nationale, Paris, Ms. Persan 169, sec. 24, fol. 191v].

The T-ruler appears from the text as an instrument devised for executing patterns that involve conic sections. It is interesting that this instrument, which looks so familiar to us as it is very similar in principle to the T-square that is ordinarily used by the architects today, was newly invented and being introduced at that particular meeting. To describe this new instrument better, it was compared to the astrolabe, an instrument peculiar to astronomy. We thus understand that the latter, even a specialized version of it, the boat astrolabe, was known more commonly than the former in those days. Hence, we can say that this ordinary looking instrument, that facilitates drawing parallel and perpendicular lines, was not actually known in the Islamic world until the turn of the fourteenth century. With its introduction about

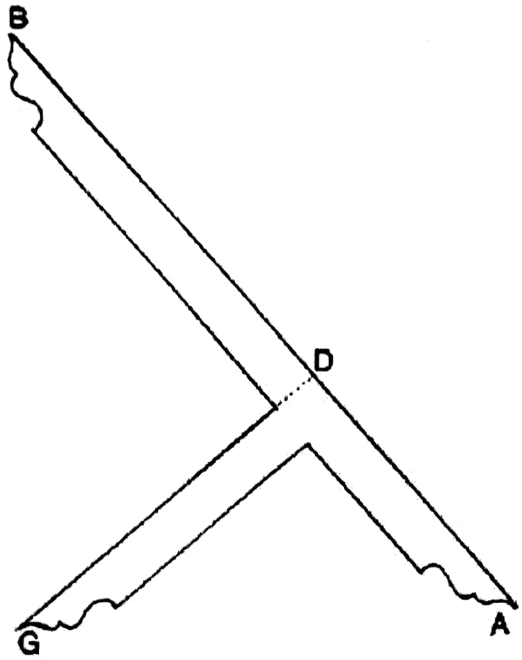


Fig. 32.4 The ABGD T-ruler described in Ms. Persan, fol. 191v. Image: author

1300, the ornamental arts and architecture appear to have gained a new impetus. So far the abundance of architectural drawings in the Islamic world after the fifteenth century was explained by the encouraging effect caused by the development of cheap paper industry during the Ilkhanid era (Necipoğlu 1995: 4–9); we can now add to it the factor of the invention of the T-square.

Al-Katibi was the name of the person who introduced the T-square at that particular meeting. The same name appears in the signature of the designer, Ali ibn Ahmad ibn Ali al-Husaini al-Katibi, of the luster tile mihrab from Imamzada Yahya at Varamin in Persia (Ritter et al. 1935: 67). Its date, 1305, suggests the attractive possibility that he and the inventor of the T-square were the same person. If this were the case, it means that a mathematician who participated in the discussions that produced *Interlocks of Figures* was also a practicing calligrapher; he thus personifies the intimate link between theory and praxis in those times.

Despite its generalized use in the future, when it was first introduced, the T-ruler was looked upon merely as a convenient tool to construct complex ornamental patterns. We see it in function again in another pattern that concern cubic equations, Construction 40 (Fig. 32.5):

> The proportion of this knot pattern is also [derived] from conics. It requires the construction of a right-angled triangle so that the altitude plus the shortest side is equal to the hypotenuse. Ibn [al-]Haytham composed a treatise on the construction of this triangle, and his construction is by means of conic sections, a hyperbola and a parabola (qaṭṭā'a-ī makhrūṭāt zayid wa mukafi). However, the objective can be achieved here with the aid of this T-ruler. According to the aforementioned preliminary, the object of our knot pattern is those four figures: "pine cones" (sanaubarī) with two right angles surrounding a right-angled equilateral and equiangular quadrangle (i.e., the square). For example, the pine cone-like quadrangles AIHK, GHMN, DMLS, and BLKO surround square KHML.
>
> Now, as angle H of the square and both [angles] of the figure are right angles, necessarily lines KG (K in the text) and HD are straight. Thus triangle AKG is

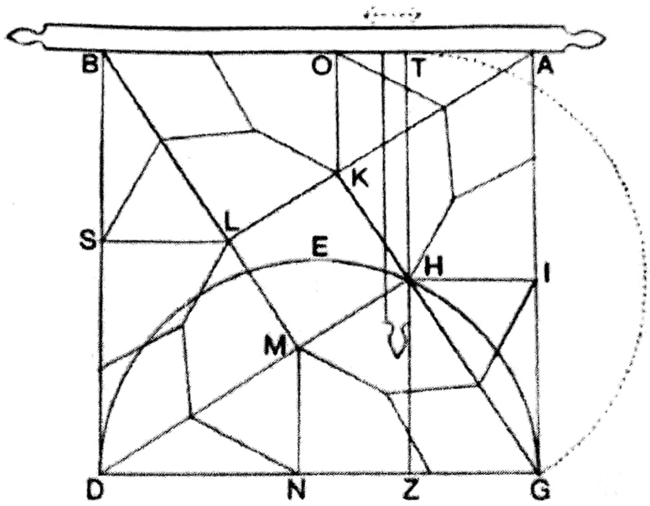


Fig. 32.5 Construction 40 of *Interlocks of Figures*. Image: author

right-angled and equal to triangle GHD. Since this triangle is right-angled, it is inscribed in a semicircle. Then point H has to be sought on arc GE (AE in the text). Subsequently, at every instance we have angle T of the ruler right-angled, its AB straight, and side AB of the [given] square and of the ruler are fixed on each other (the crucial information missing in the text can be deduced easily from the figure itself: "Give increments of sliding motion to the T-square so that it cuts the semicircle at changing positions of point H. At every instance put one arm of the compass on point H and with the other arm compare the changing lengths of HT and HG. When HT = HG, mark the point H as its required position"). God knows best [Bibliothèque Nationale, Paris, Ms. Persan 169, sec. 24, fol. 191r].

The Contributions of Omar Khayyam and al-Katibi

Although the scribe missed the crucial part of the procedure based on moving geometry, displaying again his incompetence with advanced geometrical techniques, the elegance of the restored construction indicates a high caliber mathematician behind it. When TH and HG be equal, the proof of the requirement is visible: TH + HZ = HG + HZ = GD, i.e., "the altitude plus the shortest side is equal to the hypotenuse." The scribe wrongly attributes the authorship of a treatise concerning this problem to Ibn al-Haytham. Among about 180 works of this prolific author, none answers the description. In an untitled treatise, however, Omar Khayyam describes precisely the problem concerning this special triangle, reduces its solution to a cubic equation, $x^3 - 20x^2 + 200x - 2000 = 0$, and offers two solutions by means of conic sections and one by approximation using astronomical tables (Amir-Moéz 1963). Probably the scribe was mistaken because "Khayyam" sounds similar to "Haytham," and the latter is widely known by his works on conic sections. It was evidently Omar Khayyam, also a prominent mathematician who is celebrated by his

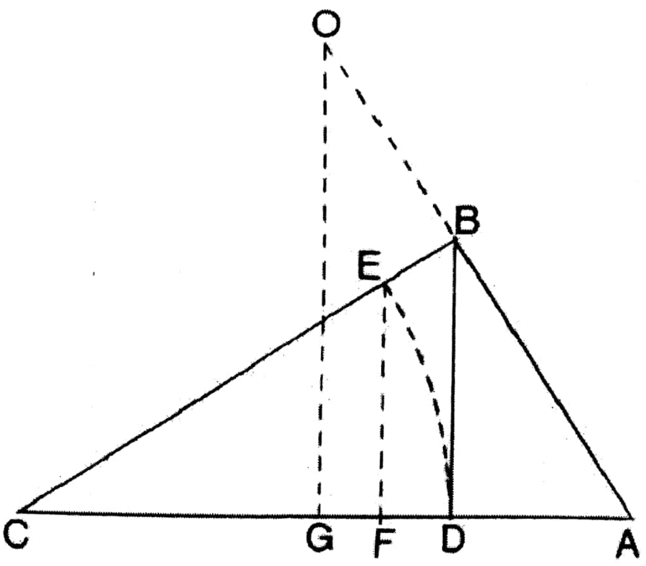


Fig. 32.6 Omar Khayyam's triangle. Image: author

works on cubic equations, who was the discoverer of this unique triangle (Fig. 32.6). When his equation is solved by modern means, angle BAC corresponds to 57.0648796°.[8]

The solution by means of moving geometry in *Interlocks of Figures* is in fact nothing but a direct translation of the problem into conic sections. The parabola is defined as the path (locus) of a point moving so that its distance from a fixed line (the directrix) is equal to its distance to a fixed point (the focus). What is achieved by the aid of T-ruler in Construction 40 answers precisely the parabola according to this definition (Fig. 32.7).

The distance of a point (H) from the directrix (side AB of the square) is set by the perpendicular arm of the T-ruler (TH), and the point's distance to the focus (G) is defined by the compass opening (HG), when the two distances be equal the point is located on the parabola. Since this particular position of point H is also located on the semicircle, the required solution thus becomes "the intersection of a parabola and a circle," as described by Omar Khayyam in his treatise.

We thus understand why al-Katibi claimed that with the T-ruler many knot patterns formed by conics could be drawn. In this example, a parabola can actually be drawn passing through points D and H by sliding the T-ruler and measuring the distances at regular intervals. The vertex of this parabola, point (O), is equidistant from the directrix and the focus. The focus and the vertex determine the axis of the parabola (GA), and the line through the focus parallel to the directrix is the *latus rectum* (GD). General equation of a parabola is $y^2 = 2px$. In our case, by assuming GD = 1, GZ = y, and ZH = x, it becomes $y^2 = 1 - 2x$. Its intersection with the circle, $x^2 + y^2 = y$, reduces the problem to the cubic equation $x^3 - 4x^2 + 6x - 2$.

This neat and simple solution suggests the authorship of a resourceful and talented mathematician, and the fact that it was based on the use of the T-ruler

[8] Assuming AC = 1, the following values are also computed for later use: AB = 0.543689, BD = 0.4563109, CB = 0.83922867, CD = 0.7044022, and AD = 0.2955977.

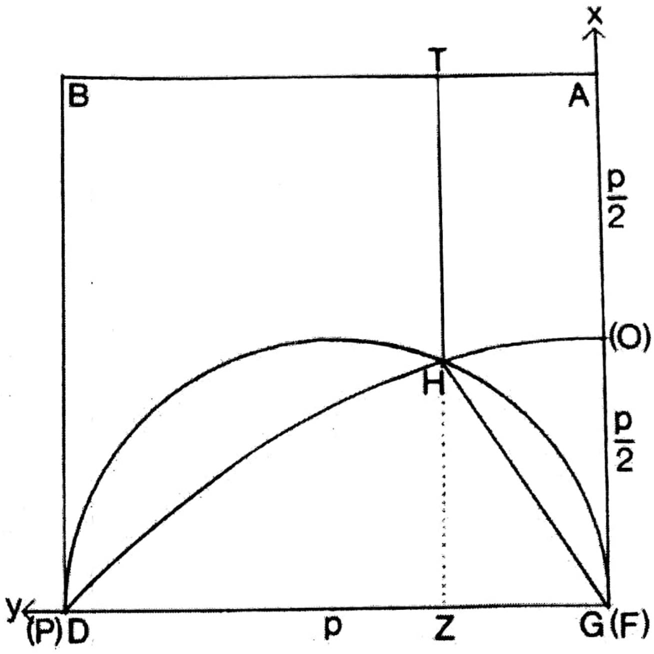


Fig. 32.7 The solution of construction 40 by means of conic sections. Image: author

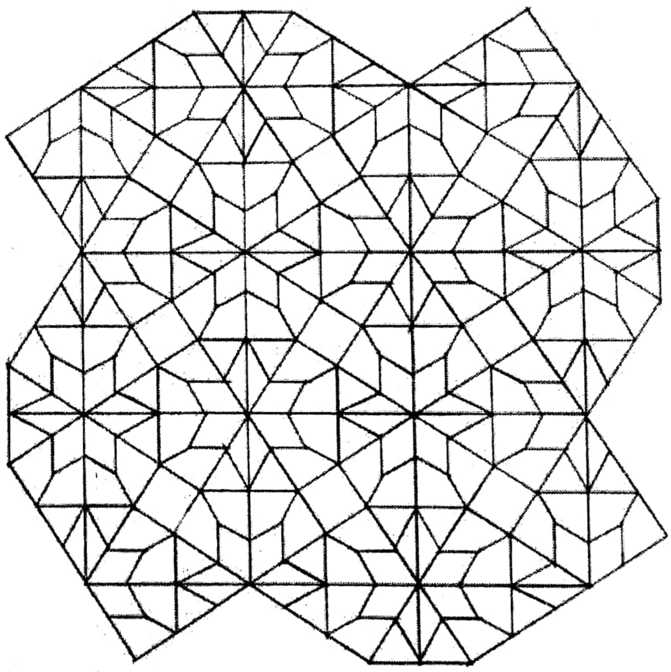
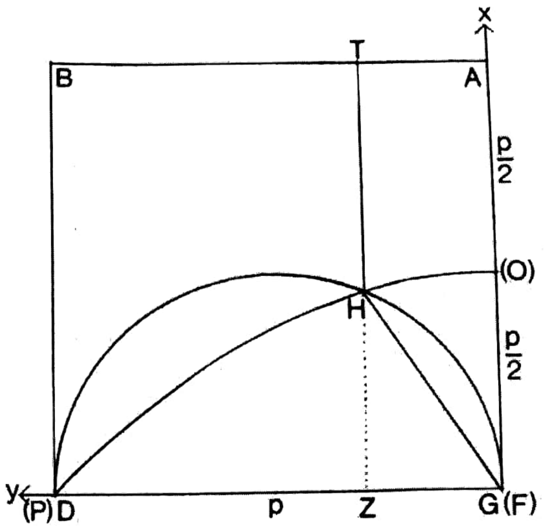


Fig. 32.8 The decorative scheme generated by construction 40. Image: author

points to its inventor, al-Katibi. If al-Katibi were both the author as well as the designer of the luster mihrab from Varamin, then the connection between mathematics and arts that he personally represents provides us a sharp insight into the milieu that created the high standard in the ornamental arts in those times. In any case, whoever the author of this solution based on moving geometry might be, he had the imagination to envisage a decorative composition as elegant as its solution (Fig. 32.8).



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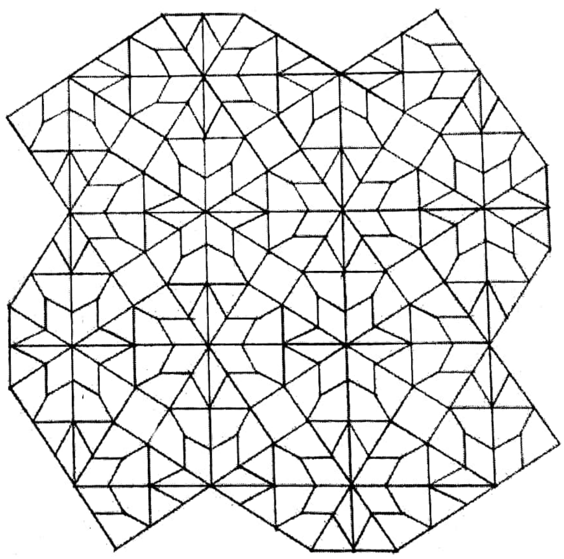

Fig. 32.8 The decorative scheme generated by construction 40. Image: author

points to its inventor, al-Katibi. If al-Katibi were both the author as well as the designer of the luster mihrab from Varamin, then the connection between mathematics and arts that he personally represents provides us a sharp insight into the milieu that created the high standard in the ornamental arts in those times. In any case, whoever the author of this solution based on moving geometry might be, he had the imagination to envisage a decorative composition as elegant as its solution (Fig. 32.8).

Omar Khayyam's Triangle and the North Dome Chamber

This decorative scheme is not the only outcome of Omar Khayyam's triangle. He says:

> A triangle with mentioned properties is very useful in problems similar to this one. This triangle has other properties. We shall mention some of them so that whoever studies this paper can benefit from it in similar problems (Amir-Moéz 1963: 326).

A more attentive study of this triangle reveals indeed some very interesting properties (Fig. 32.6): between the hypotenuse AC and the shortest side AB, CB becomes the geometric mean, CD the harmonic mean, and GO, the perpendicular on the midpoint of AC, the arithmetic mean.[9] Then, AC:GO::CD:AB, which Greeks called "the musical proportion" and judged most perfect. Simpler numerical versions of this proportion, such as 12:9::8:6, are known to have been used in Renaissance architecture. In the hands of Omar Khayyam this proportion attains a mathematical complexity with its irrational magnitudes obtained by means of cubic equations. The richly interrelated proportions embodied in Omar Khayyam's triangle present themselves provocatively as tools suitable for architectural application.

The North Dome Chamber in the Friday mosque of Isfahan has always impressed its visitors with its mature proportions. Eric Schroeder has judged it as marrying genius and tradition more elegantly than many other famous domed structures (Schroeder 1938–1939: III, 1005). Its foundation is dated 1088–1089. It was the time when Omar Khayyam was enjoying high prestige as the leading astronomer in Isfahan who had already published his major works on geometry and algebra, and whose reformed calendar was lately adopted by royal decree.

If Omar Khyyam's triangle is superimposed over a drawing of the North Dome so that the hypotenuse corresponds to the span, we perceive that the generating force behind that astoundingly powerful space is the musical proportion contained in this singularly unique triangle (Fig. 32.9 and Table 32.1). The geometric scheme is my own, but its very close agreement with actual dimensions gives it credibility and suggests strongly that Omar Khayyam, one of the greatest intellects the Islamic world had produced, was actually the designer of one of the greatest accomplishments of Islamic architecture.[10]

[9] The following values are computed in addition to the ones in n. 16:GO = 0.7718445, CF = 0.5911954 = CB^2.

[10] For my detailed argument on this point, see (Özdural 1998: 699–715). The photogrammetric drawing and the dimensions obtained from Rassad Survey Company, "Masjed-e Jame'Esfahan" (paper presented to the International Committee for Architectural Photogrammetry at the Symposium on the Photogrammetric Survey of Ancient Monuments, Athens, 1974, pl. 13), are published in ibid., Fig. 32.5.

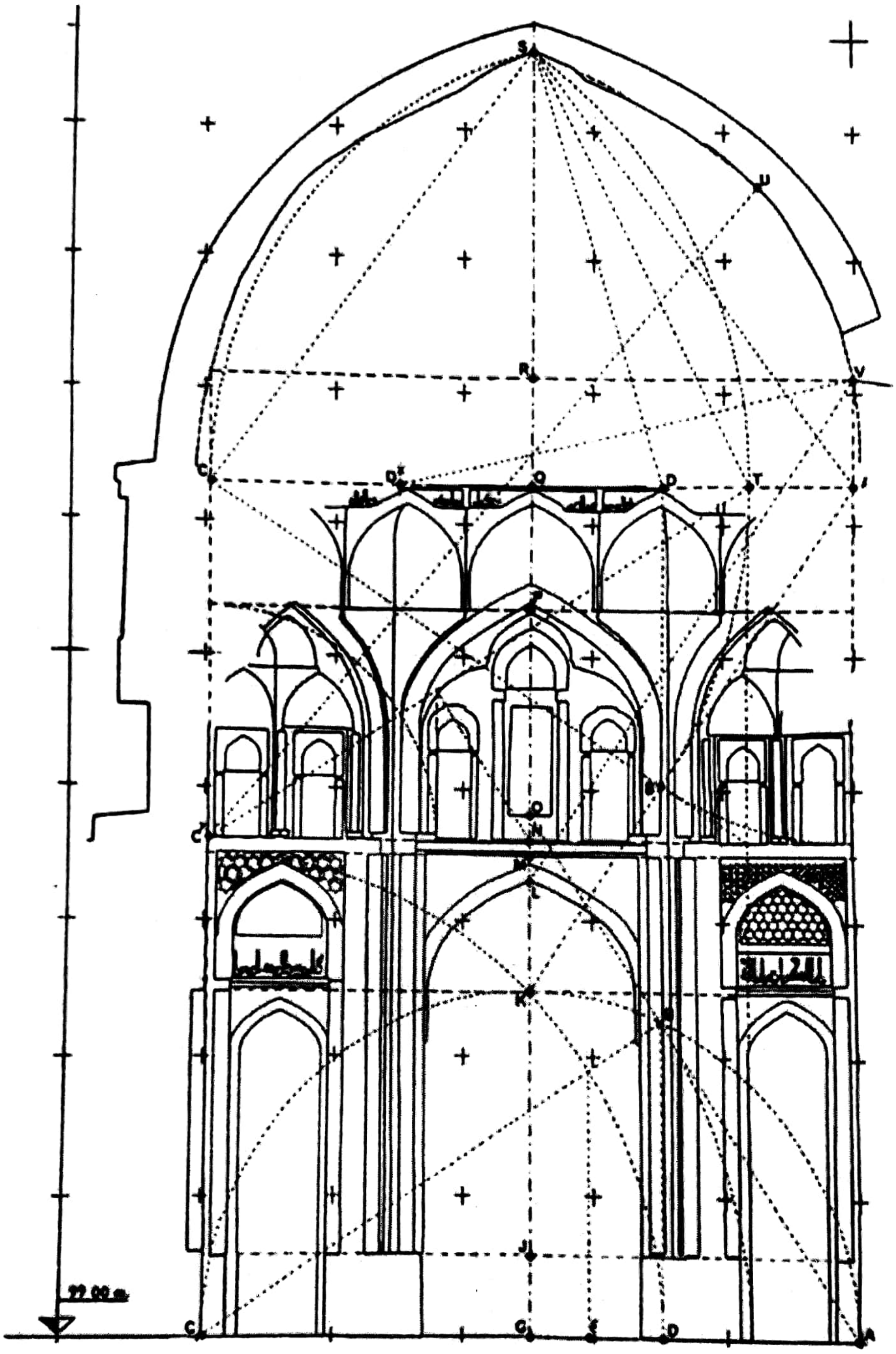

Fig. 32.9 The geometric scheme generated by Omar Khayyam's triangle. Image: author

Biography Alpay Özdural (1944–2003) was Associate Professor at Eastern Mediterranean University, Faculty of Architecture in North Cyprus, where he taught design and history of architecture courses. By profession he was an architect specialized in restoration and preservation of historic monuments and sites. His other fields of interest were architectural photogrammetry, history of mathematics, and historical metrology. The last 10 years of his life he concentrated his efforts on muqarnas, a type of three-dimensional geometric decoration peculiar